Comments on Manipulability Measure in Redundant Planar Manipulators

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Outline

1. Introduction
2. Manipulability Measure
3. Eigen-analysis of M
4. Manipulability Driven Control
5. Results
6. Conclusions
Kinematics

- Kinematics control of the redundant manipulators;
- The classical solution is based on the calculation of the pseudo-inverse Jacobian matrix;
- Problem with infinite solutions;
- High computational coast.
Manipulability

- Manipulability is the ability of robotic mechanisms in positioning and orienting the end-effectors;
- The manipulability varies as a function of the manipulator configuration;
- The manipulability ellipsoid is defined by Jacobian matrix.
Motivation

Some manipulability measures applications
- To evaluate and analyze a control system;
- To optimize the system performance;
- To planning the trajectories.

Our Motivation is to analyze the manipulability matrix of the planar redundant manipulator, to reduce its joint angle variations (smooth movements).
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Manipulability Measure

Manipulator Kinematics

- Planar manipulator;
- End-effector’s position:

\[
x(\theta_1, \ldots, \theta_n) = \sum_{i=1}^{n} L_i \cos \left( \sum_{j=1}^{i} \theta_j \right)
\]

\[
y(\theta_1, \ldots, \theta_n) = \sum_{i=1}^{n} L_i \sin \left( \sum_{j=1}^{i} \theta_j \right)
\] (1)
Manipulability Measure

Manipulator Kinematics

- Differentiating (1), and putting it on a vectorial representation:

\[
\begin{bmatrix}
\dot{x}(t, \theta_1, \ldots, \theta_n) \\
\dot{y}(t, \theta_1, \ldots, \theta_n)
\end{bmatrix} = J \dot{\theta}
\]  \hspace{1cm} (2)

- The elements of the Jacobian matrix \( J \), are:

\[
J_{1c} = \sum_{m=c}^{n} -L_m \sin \left( \sum_{j=1}^{m} \theta_j \right) ; \quad J_{2c} = \sum_{m=c}^{n} L_m \cos \left( \sum_{j=1}^{m} \theta_j \right)
\]  \hspace{1cm} (3)
Manipulability Measure

- The manipulability measure:
  \[ m = \sqrt{\det(J J^t)} \]  
  \hspace{1cm} (4)

- The manipulability matrix:
  \[ M = J J^t \]  
  \hspace{1cm} (5)

- The matrix \( M \) is symmetric positive definite matrix used to obtain the manipulability ellipsoid;

- The manipulability ellipsoid is defined as the set of all cartesian velocities corresponding to \( d\theta/dt \).

  \[ \frac{d\theta}{dt} \leq 1 \]  
  \hspace{1cm} (6)
Manipulability Measure

- Its axes length are the eigenvalues of $M$ and the eigenvectors give its orientation;
- The value of $m$ is proportional to the ellipsoid area;
- The matrix $M$, for a 2D manipulator:

$$
M = \begin{bmatrix}
\sum_{i=1}^{n} \left( \frac{\partial x}{\partial \theta_i} \right)^2 & \sum_{i=1}^{n} \frac{\partial x}{\partial \theta_i} \frac{\partial y}{\partial \theta_i} \\
\sum_{i=1}^{n} \frac{\partial x}{\partial \theta_i} \frac{\partial y}{\partial \theta_i} & \sum_{i=1}^{n} \left( \frac{\partial y}{\partial \theta_i} \right)^2
\end{bmatrix}
$$

(7)

- The determinant of the matrix $M$:

$$
\det(M) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial x}{\partial \theta_i} \frac{\partial x}{\partial \theta_i} \frac{\partial y}{\partial \theta_j} \frac{\partial y}{\partial \theta_j} - \frac{\partial x}{\partial \theta_i} \frac{\partial y}{\partial \theta_i} \frac{\partial x}{\partial \theta_j} \frac{\partial y}{\partial \theta_j}
$$

(8)
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Proposition

The manipulability measure $m$ does not depend on the first joint angle.

- Differentiating (1)

\[
\frac{\partial x}{\partial \theta_i} = - \sum_{t=i}^{n} L_t \sin \left( \sum_{p=1}^{t} \theta_p \right) = x_i
\]

\[
\frac{\partial y}{\partial \theta_i} = \sum_{t=i}^{n} L_t \cos \left( \sum_{p=1}^{t} \theta_p \right) = y_i
\]
Proof 1

- Differentiating (9) with respect to the first joint angle $\theta_1$:

$$\frac{\partial x_i}{\partial \theta_1} = -\sum_{t=i}^{n} L_t \cos \left( \sum_{p=1}^{t} \theta_p \right) = -y_i$$  \hspace{1cm} (10)

$$\frac{\partial y_i}{\partial \theta_1} = -\sum_{t=i}^{n} L_t \sin \left( \sum_{p=1}^{t} \theta_p \right) = x_i$$

- The angle $\theta_1$ appears in all terms of the summation;
- Substituting the notation in the determinant of $M$:

$$|M| = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j y_i y_j - x_i y_i x_j y_j$$  \hspace{1cm} (11)
Proof 1

- Differentiating (11) with respect to $\theta_1$:

$$\frac{\partial |M|}{\partial \theta_1} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial x_i}{\partial \theta_1} x_i y_j y_j + x_i \frac{\partial x_i}{\partial \theta_1} y_j y_j + x_i x_i \frac{\partial y_j}{\partial \theta_1} y_j + x_i x_i y_j \frac{\partial y_j}{\partial \theta_1} ight) - \frac{\partial x_i}{\partial \theta_1} y_i x_i y_j - x_i \frac{\partial y_i}{\partial \theta_1} x_j y_j - x_i y_i \frac{\partial x_j}{\partial \theta_1} y_j - x_i y_i x_j \frac{\partial y_j}{\partial \theta_1}$$

(12)

- Using (10) in (12):

$$\frac{\partial |M|}{\partial \theta_1} = \sum_{i=1}^{n} \sum_{j=1}^{n} -y_i x_i y_j y_j + x_i x_i x_j y_j + y_i y_i x_j y_j - x_i y_i x_j x_j$$

(13)
Proof 1

- Separating the summations:

\[
\frac{\partial |\mathbf{M}|}{\partial \theta_1} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_i x_j y_j + y_i y_i x_j y_j
\]

\[
- \sum_{i=1}^{n} \sum_{j=1}^{n} y_i x_i y_j y_j + x_i y_i x_j x_j
\]

(14)

- Swapping the indexes in (14):

\[
\frac{\partial |\mathbf{M}|}{\partial \theta_1} = 0
\]
Proof 2

Proposition

The eigenvalues does not depend on the first joint angle, as long as their product (determinant of M).

- Adopting:

\[
M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}
\]  
(15)

- The eigenvalues of \( M \) satisfies the characteristic equation:

\[
\lambda^2 - \lambda(a + b) + |M| = 0
\]
(16)
Proof 2

Differentiating both sides of the (16) with respect to $\theta_1$:

$$2\lambda \frac{\partial \lambda}{\partial \theta_1} - \frac{\partial \lambda}{\partial \theta_1} (a + b) - \lambda \left( \frac{\partial a}{\partial \theta_1} + \frac{\partial b}{\partial \theta_1} \right) + \frac{\partial |M|}{\partial \theta_1} = 0 \quad (17)$$

Determinant of $M$ does not depend of $\theta_1$, so:

$$\frac{\partial \lambda}{\partial \theta_1} (2\lambda - (a + b)) = \lambda \left( \frac{\partial a}{\partial \theta_1} + \frac{\partial b}{\partial \theta_1} \right) \quad (18)$$
Proof 2

Examining the term in the right side of the (18):

\[
\frac{\partial a}{\partial \theta_1} + \frac{\partial b}{\partial \theta_1} = \frac{\partial \left( \sum_{i=1}^{n} x_i^2 \right)}{\partial \theta_1} + \frac{\partial \left( \sum_{i=1}^{n} y_i^2 \right)}{\partial \theta_1} \\
= \frac{\partial \left( \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \right)}{\partial \theta_1} \\
= \frac{\partial \left( \sum_{i=1}^{n} x_i^2 + y_i^2 \right)}{\partial \theta_1} \tag{19}
\]
Proof 2

- Solving the derivative:

\[
\frac{\partial a}{\partial \theta_1} + \frac{\partial b}{\partial \theta_1} = \sum_{i=1}^{n} 2x_i \frac{\partial x_i}{\partial \theta_1} + 2y_i \frac{\partial y_i}{\partial \theta_1}
\]

\[
= \sum_{i=1}^{n} -2x_i y_i + 2y_i x_i = 0 \quad (20)
\]

- Substituting (20) in (18):

\[
\frac{\partial \lambda}{\partial \theta_1} (2\lambda - (a + b)) = 0 \quad (21)
\]

- The right side of (18) is zero and in general \(2\lambda \neq a + b\), therefore, the derivative of \(\lambda\) is always equal to zero.
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Manipulability Ellipsoid

Manipulator 1
Manipulability Ellipsoid

Manipulator 2
Proposed Control Law

- Kinematics control based on the Jacobian:
  \[
  d\theta = J^+ \Delta p dt
  \]  
  (22)

- The position displacement:
  \[
  \Delta p = e = d - x
  \]  
  (23)

- The modified position displacement:
  \[
  \Delta p = e + k \| e \| u
  \]  
  (24)

- \( u \) is a unitary vector pointing in the direction of the larger eigenvector of \( M \);
- The manipulator avoid paths of low manipulability.
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Classical Control (Resolved Rate)
Manipulability Driven Control
## Comparisons

### Average values of $d\theta$

<table>
<thead>
<tr>
<th>Controllers</th>
<th>$d\theta_1$</th>
<th>$d\theta_2$</th>
<th>$d\theta_3$</th>
<th>$d\theta_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td>0.0208</td>
<td>0.0047</td>
<td>0.0083</td>
<td>0.0303</td>
</tr>
<tr>
<td>Manipulability driven</td>
<td>0.0060</td>
<td>0.0006</td>
<td>0.0011</td>
<td>0.0041</td>
</tr>
</tbody>
</table>
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Conclusions

- The manipulability measure and the eigenvalues of $M$ is independent of the first joint angle of the manipulator;
- The eigenvectors of the manipulability matrix depend on the first joint angle;
- Manipulability Driven Control Law was proposed;
- The results show that the control keeps the manipulator on configurations with bigger manipulability;
- The control law can be extended for more complex controls.
Acknowledgments

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