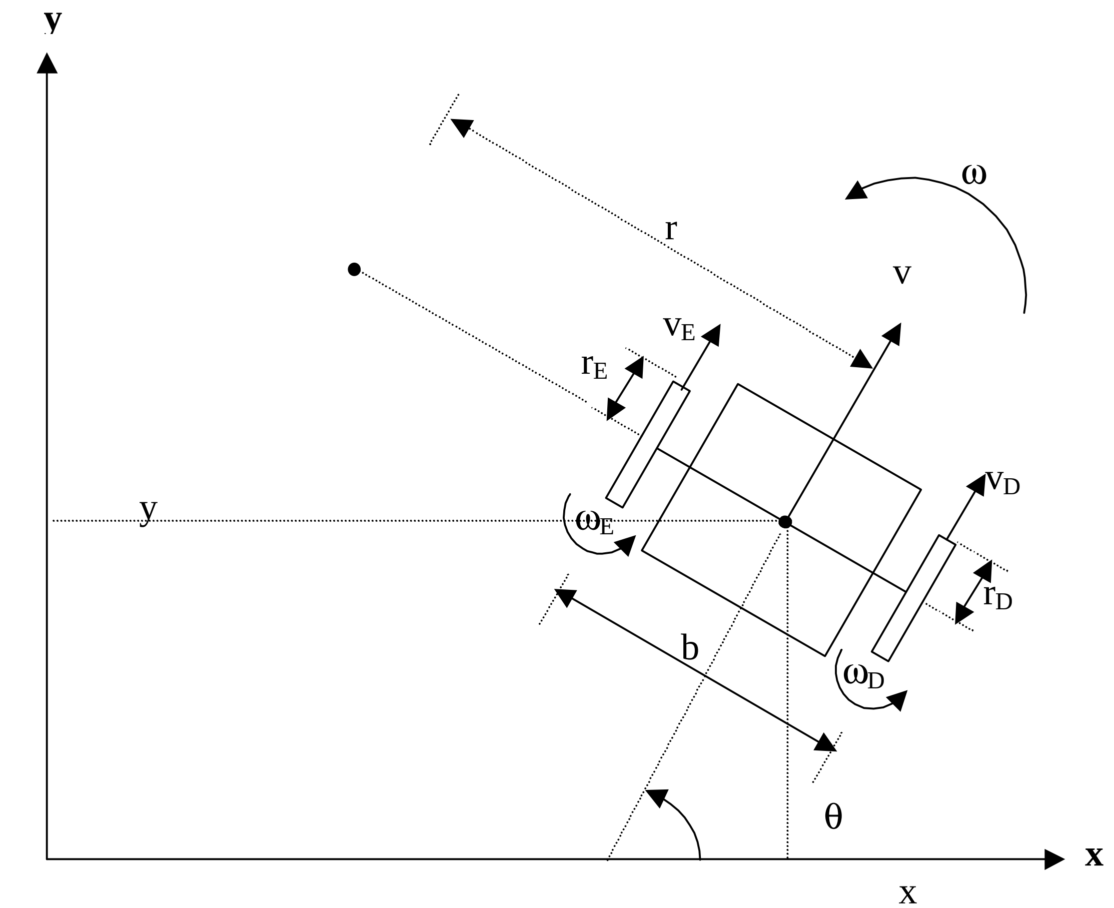
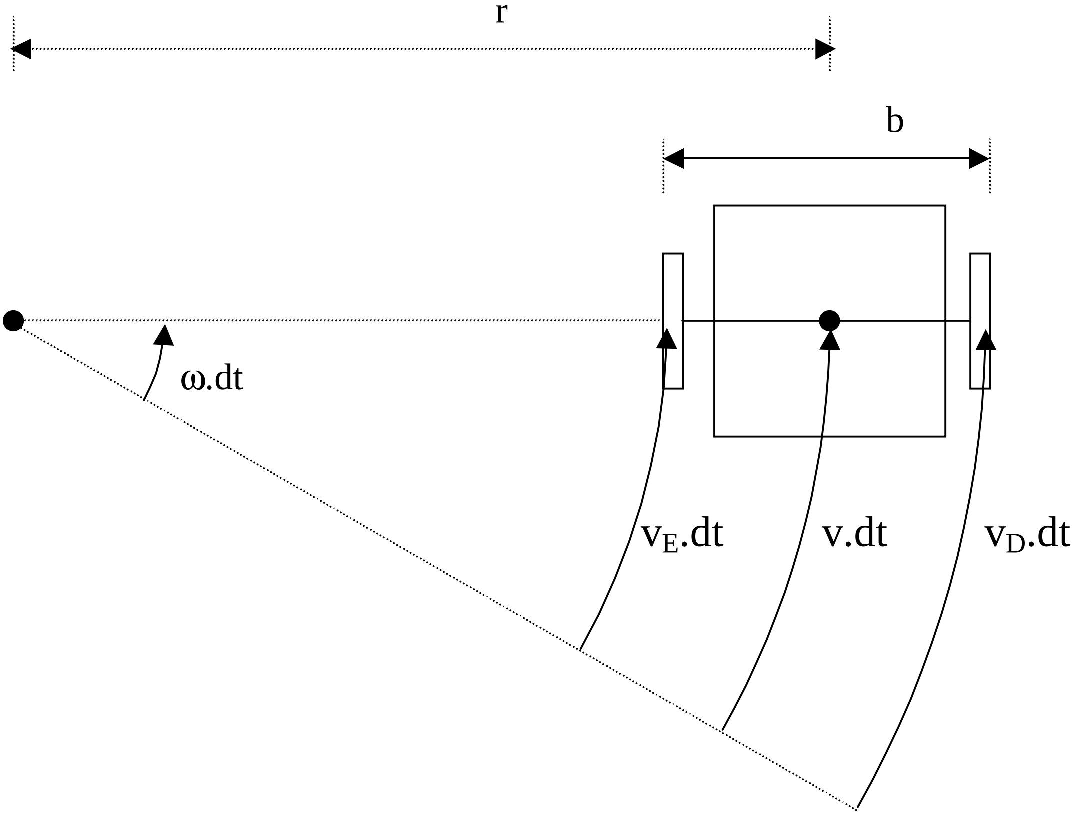
MODELO CINEMÁTICO DE UM ROBÔ MÓVEL



(x,y) =	Posição do referencial fixo no robô em relação ao referencial fixo no espaço de trabalho.
θ =	Ângulo de orientação do robô em relação ao referencial fixo no espaço de trabalho.
b =	Comprimento do eixo.
r =	Raio de giro do robô.
r_D (r_E) =	Raio da roda direita (esquerda)
ω =	Velocidade angular do robô.
ω_D (ω_E) =	Velocidade angular da roda direita (esquerda).
v =	Velocidade linear do robô $\Rightarrow v = \omega.r$
v_D (v_E) =	Velocidade linear da borda da roda direita (esquerda).
	$\Rightarrow v_D = \omega_D.r_D$
	$\Rightarrow v_E = \omega_E.r_E$

Para movimentos infinitesimais:



$$\begin{cases} v_D.dt = \omega(r+b/2)dt \\ v_E.dt = \omega(r-b/2)dt \end{cases} \Rightarrow \begin{cases} v_D+v_E = \omega_D.r_D+\omega_E.r_E = 2.\omega.r = 2.v \\ v_D-v_E = \omega_D.r_D-\omega_E.r_E = \omega.b \end{cases}$$

$$\Rightarrow \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} (r_D/2) & (r_E/2) \\ (r_D/b) & -(r_E/b) \end{bmatrix} . \begin{bmatrix} \omega_D \\ \omega_E \end{bmatrix} \Rightarrow \mathbf{V} = {}^{\mathbf{V}}\mathbf{T}_{\mathbf{W}}.\mathbf{W}$$

onde: $\mathbf{V} = [v \quad \omega]^T$ $\quad \mathbf{W} = [\omega_D \quad \omega_E]^T$ $\quad {}^{\mathbf{V}}\mathbf{T}_{\mathbf{W}} = \begin{bmatrix} (r_D/2) & (r_E/2) \\ (r_D/b) & -(r_E/b) \end{bmatrix}$

$\Rightarrow \mathbf{W} = ({}^{\mathbf{V}}\mathbf{T}_{\mathbf{W}})^{-1}.\mathbf{V} = {}^{\mathbf{W}}\mathbf{T}_{\mathbf{V}}.\mathbf{V}$ $\qquad$ onde, ${}^{\mathbf{W}}\mathbf{T}_{\mathbf{V}} = ({}^{\mathbf{V}}\mathbf{T}_{\mathbf{W}})^{-1}$

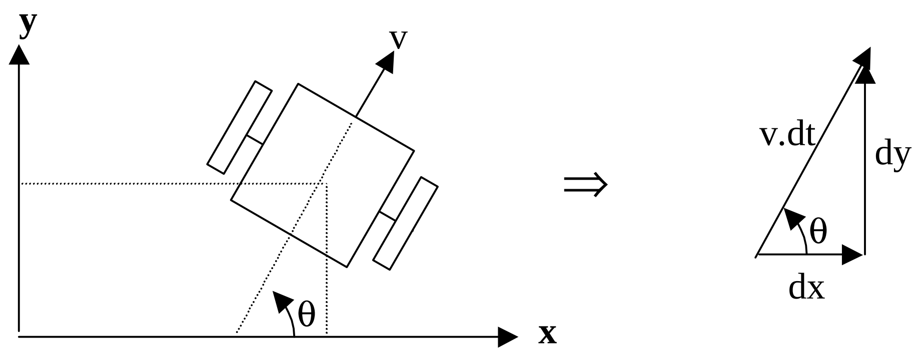
Relação entre velocidades das rodas (ω_E/ω_D) para mover-se com raio de giro r:

$(2.\omega.r)/(\omega.b) = (2.r/b) = (\omega_D.r_D+\omega_E.r_E)/(\omega_D.r_D-\omega_E.r_E)$

$\Rightarrow (\omega_E/\omega_D) = [(r-b/2).r_D]/[(r+b/2).r_E]$

- **Restrições não holonômicas**:

  - Restrições não holonômicas atuam nas velocidades do robô.
  - Devido ao atrito das rodas, o robô não pode se deslocar lateralmente (na direção do eixo).
  - A velocidade linear sempre aponta na direção definida pela orientação θ do robô.



Para deslocamentos infinitesimais:

$dx = v.dt.\cos\theta \Rightarrow dx/dt = x' = v.\cos\theta$
$dy = v.dt.\text{sen}\theta \quad \Rightarrow dy/dt = y' = v.\text{sen}\theta$
$d\theta = \omega.dt \qquad \Rightarrow d\theta/dt = \theta' = \omega$

$$\Rightarrow \begin{cases} (dy/dx) = \tan(\theta) = \text{sen}\theta/\cos\theta \\ v.dt = dx.\cos\theta + dy.\text{sen}\theta \end{cases} \quad \Rightarrow \quad dy.\cos\theta - dx.\text{sen}\theta = 0$$

$$\Rightarrow \begin{bmatrix} \cos\theta & \text{sen}\theta \\ -\text{sen}\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

- **Modelo cinemático**:

Definindo o <u>Vetor de Variáveis de Configuração</u>, ou, simplesmente, configuração $\mathbf{q} = [x \;\; y \;\; \theta]^T$, lembrando que:

$$x' = v.\cos\theta; \qquad y' = v.\text{sen}\theta; \qquad \theta' = \omega;$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \text{sen}\theta & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} \quad \text{onde,} \begin{bmatrix} \cos\theta & 0 \\ \text{sen}\theta & 0 \\ 0 & 1 \end{bmatrix} = {}^{q}T_{V}$$

$$\Rightarrow \quad \mathbf{q'} = {}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}.\mathbf{V}$$
$$\Rightarrow \quad \mathbf{q'} = {}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}.{}^{\mathbf{V}}\mathbf{T}_{\mathbf{W}}.\mathbf{W}$$

- **Propriedades da matriz $^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}$**:

i) $$({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T.({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}) = \begin{bmatrix} \cos\theta & \text{sen}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & 0 \\ \text{sen}\theta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow ({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T.\mathbf{q'} = ({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T.\,({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).\mathbf{V} = \mathbf{V} \qquad \Rightarrow \mathbf{V} = ({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T.\mathbf{q'}$$

ii) $$({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T = \begin{bmatrix} \cos\theta & 0 \\ \text{sen}\theta & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & \text{sen}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad ({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T = \begin{bmatrix} \cos^2\theta & \cos\theta.\text{sen}\theta & 0 \\ \cos\theta.\text{sen}\theta & \text{sen}^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

onde, a matriz $({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T$ é singular. Apesar disto,

$$\mathbf{q'} = ({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T.\mathbf{q'}$$

visto que, de (i): $({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}})^T.\mathbf{q'} = ({}^{\mathbf{q}}\mathbf{T}_{\mathbf{V}}).\mathbf{V} = \mathbf{q'}$

iii) $({}^{q}\mathbf{T}_V)^T = ({}^{q}\mathbf{T}_V)^+$

Ou seja, $({}^{q}\mathbf{T}_V)^T$ é a matriz pseudo-inversa de ${}^{q}\mathbf{T}_V$, o que decorre diretamente de (i) e (ii).

iv) $$({}^{q}\mathbf{T}_V)^T.({}^{q}\mathbf{T}_V)' = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} -\operatorname{sen}\theta & 0 \\ \cos\theta & 0 \\ 0 & 0 \end{bmatrix} .\theta' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{q''} = (\mathbf{q'})' = (({}^{q}\mathbf{T}_V).\mathbf{V})' = {}^{q}\mathbf{T}_V.\mathbf{V'} + ({}^{q}\mathbf{T}_V)'.\mathbf{V} \Rightarrow$$

$$\Rightarrow ({}^{q}\mathbf{T}_V)^T.\ \mathbf{q''} = ({}^{q}\mathbf{T}_V)^T.{}^{q}\mathbf{T}_V.\mathbf{V''} + ({}^{q}\mathbf{T}_V)^T.({}^{q}\mathbf{T}_V)'.\mathbf{V}$$

$$\Rightarrow ({}^{q}\mathbf{T}_V)^T.\ \mathbf{q''} = \mathbf{V''}$$

v) $$({}^{q}\mathbf{T}_V).(({}^{q}\mathbf{T}_V)^T)' = \begin{bmatrix} \cos\theta & 0 \\ \operatorname{sen}\theta & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} .\theta'$$

$$\Rightarrow ({}^{q}\mathbf{T}_V).(({}^{q}\mathbf{T}_V)^T)' = \begin{bmatrix} -\cos\theta.\operatorname{sen}\theta & \cos^2\theta & 0 \\ -\operatorname{sen}^2\theta & \cos\theta.\operatorname{sen}\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} .\theta'$$

onde, a matriz $({}^{q}\mathbf{T}_V).(({}^{q}\mathbf{T}_V)^T)'$ é singular.

$$\Rightarrow \mathbf{V'} = (({}^{q}\mathbf{T}_V)^T.\mathbf{q'})' = ({}^{q}\mathbf{T}_V)^T.\mathbf{q''} + (({}^{q}\mathbf{T}_V)^T)'.\mathbf{q'}$$

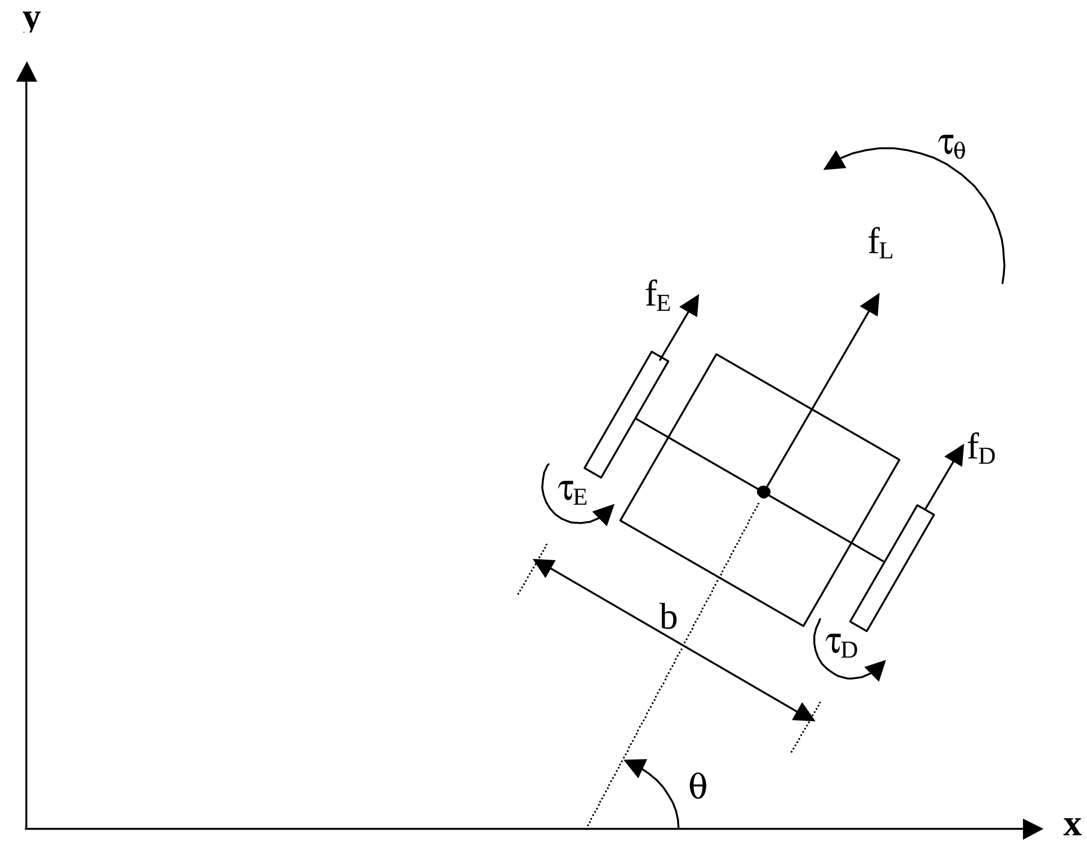
$$\Rightarrow ({}^{q}\mathbf{T}_V).\mathbf{V'} = ({}^{q}\mathbf{T}_V).(({}^{q}\mathbf{T}_V)^T.\mathbf{q'})'$$
$$= ({}^{q}\mathbf{T}_V).({}^{q}\mathbf{T}_V)^T.\mathbf{q''} + ({}^{q}\mathbf{T}_V).(({}^{q}\mathbf{T}_V)^T)'.\mathbf{q'}$$

Mas, $(({}^{q}\mathbf{T}_V)^T)'.\mathbf{q'} = \theta'. \begin{bmatrix} -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow ({}^{q}\mathbf{T}_V).\mathbf{V'} = ({}^{q}\mathbf{T}_V).({}^{q}\mathbf{T}_V)^T.\mathbf{q''}$$

multiplicando por $({}^{q}\mathbf{T}_V)^T \Rightarrow \quad \mathbf{V'} = ({}^{q}\mathbf{T}_V)^T.\mathbf{q''}$

<u>ESFORÇOS ESTÁTICOS</u>



f_L =	Força resultante no robô.
f_D (f_E) =	força na borda da roda direita (esquerda).
τ_θ =	Torque resultante no robô.
τ_D (τ_E) =	Torque na roda direita (esquerda).

$\Rightarrow \tau_D = f_D.r_D \qquad \Rightarrow f_D = \tau_D/r_D$

$\Rightarrow \tau_E = f_E.r_E \qquad \Rightarrow f_E = \tau_E/r_E$

- **Esforços resultantes no robô, $\mathbf{F_V} = [f_L \quad \tau_\theta]^T$:**

$$\begin{cases} f_L = f_D + f_E = (\tau_D/r_D) + (\tau_E/r_E) \\ \\ \tau_\theta = (f_D.b/2) - (f_E.b/2) = (\tau_D.b/2r_D) - (\tau_E.b/2r_E) \end{cases}$$

$$\Rightarrow \begin{bmatrix} f_L \\ \tau_\theta \end{bmatrix} = \begin{bmatrix} (1/r_D) & (1/r_E) \\ (b/2r_D) & (-b/2r_E) \end{bmatrix} . \begin{bmatrix} \tau_D \\ \tau_E \end{bmatrix} \Rightarrow \mathbf{F_V} = (({}^{\mathbf{V}}\mathbf{T_W})^T)^{-1}.\boldsymbol{\tau}$$

onde, $\mathbf{F_V} = [f_L \quad \tau_\theta]^T \quad \boldsymbol{\tau} = [\tau_D \quad \tau_E]^T$

$$\begin{bmatrix} (1/r_D) & (1/r_E) \\ (b/2r_D) & (-b/2r_E) \end{bmatrix} = ({}^{\mathbf{W}}\mathbf{T_V})^T = (({}^{\mathbf{V}}\mathbf{T_W})^T)^{-1} \quad \Rightarrow \boldsymbol{\tau} = ({}^{\mathbf{V}}\mathbf{T_W})^T.\ \mathbf{F_V}$$

- **Esforços em espaço de configuração, $\mathbf{F} = [f_x \ \ f_y \ \ \tau_\theta]^T$:**

$$\begin{cases} f_x = f_L.\cos\theta \\ f_y = f_L.\text{sen}\theta \\ \tau_\theta = \tau_\theta \end{cases} \Rightarrow \begin{bmatrix} f_x \\ f_y \\ \tau_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \text{sen}\theta & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} f_L \\ \tau_\theta \end{bmatrix}$$

$$\Rightarrow \mathbf{F} = {}^{\mathbf{q}}\mathbf{T_V}.\mathbf{F_V} \Rightarrow ({}^{\mathbf{q}}\mathbf{T_V})^T.\mathbf{F} = ({}^{\mathbf{q}}\mathbf{T_V})^T.{}^{\mathbf{q}}\mathbf{T_V}.\mathbf{F_V} \Rightarrow \mathbf{F_V} = ({}^{\mathbf{q}}\mathbf{T_V})^T.\mathbf{F}$$

$$\Rightarrow \begin{cases} \mathbf{F} = {}^{\mathbf{q}}\mathbf{T_V}.(({}^{\mathbf{V}}\mathbf{T_W})^T)^{-1}.\boldsymbol{\tau} \\ \boldsymbol{\tau} = ({}^{\mathbf{V}}\mathbf{T_W})^T.({}^{\mathbf{q}}\mathbf{T_V})^T.\mathbf{F} \end{cases}$$

- **Conservação de Potência**:

Sejam: potência em espaço de configuração = $P_q = \mathbf{F}^T.\mathbf{q'}$
potência em espaço de atuadores = $P_W = \boldsymbol{\tau}^T.\mathbf{W}$

$$\Rightarrow P_q = \mathbf{F}^T.\mathbf{q'} = [{}^{\mathbf{q}}\mathbf{T_V}.(({}^{\mathbf{V}}\mathbf{T_W})^T)^{-1}.\boldsymbol{\tau}]^T.[{}^{\mathbf{q}}\mathbf{T_V}.{}^{\mathbf{V}}\mathbf{T_W}.\mathbf{W}]$$

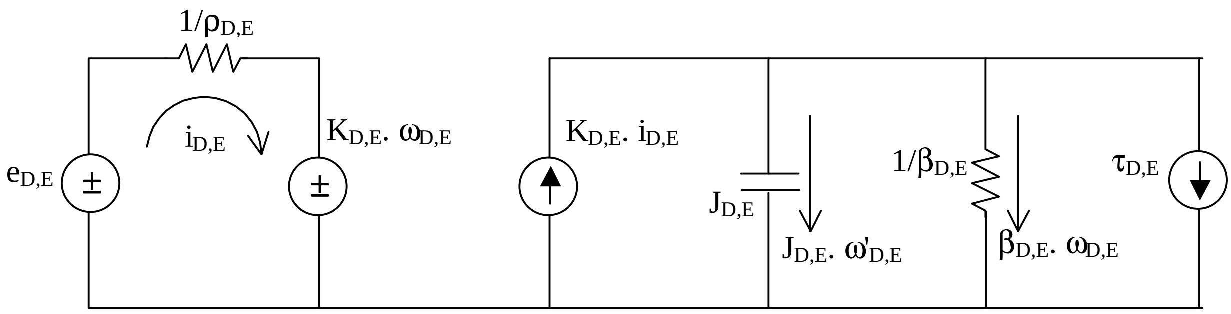
$$= \boldsymbol{\tau}^T.({}^{\mathbf{V}}\mathbf{T_W})^{-1}({}^{\mathbf{q}}\mathbf{T_V})^T.{}^{\mathbf{q}}\mathbf{T_V}.{}^{\mathbf{V}}\mathbf{T_W}.\mathbf{W} = \boldsymbol{\tau}^T.({}^{\mathbf{V}}\mathbf{T_W})^{-1}.{}^{\mathbf{V}}\mathbf{T_W}.\mathbf{W} = \boldsymbol{\tau}^T.\mathbf{W}$$

$$= P_W \quad \Rightarrow P_q = P_W$$

MODELO DINÂMICO DE UM ROBÔ MÓVEL

- **Robô Móvel**: duas rodas, com acionamento diferencial através de motores CC.

- **Dinâmica de Atuadores (Motores CC)**:



$e_{D,E}$ =	Tensão de armadura do motor (Direito, Esquerdo).
$i_{D,E}$ =	Corrente de armadura do motor (Direito, Esquerdo).
$1/\rho_{D,E}$ =	Resistência de armadura (Direita, Esquerda).
$K_{D,E}$ =	Constante de força contra-eletromotriz / torque do motor (Direito, Esquerdo).
$J_{D,E}$ =	Momento de inércia do rotor (Direito, Esquerdo).
$\beta_{D,E}$ =	Coeficiente de atrito do motor (Direito, Esquerdo).
$\tau_{D,E}$ =	Carga mecânica no motor (Direito, Esquerdo).

Equação Elétrica do Motor:

$$\mathbf{E} = \rho^{-1}.\mathbf{i} + \mathbf{K_W}.\mathbf{W} \quad \Rightarrow \quad \mathbf{i} = -\rho.\mathbf{K_W}.\mathbf{W} + \rho.\mathbf{E}$$

onde: $\mathbf{E} = [e_D \;\; e_E]^T$ $\quad \mathbf{i} = [i_D \;\; i_E]^T$ $\quad \mathbf{W} = [\omega_D \;\; \omega_E]^T$

$$\rho = \begin{bmatrix} \rho_D & 0 \\ 0 & \rho_E \end{bmatrix} \qquad \mathbf{K_W} = \begin{bmatrix} K_D & 0 \\ 0 & K_E \end{bmatrix}$$

Equação Mecânica do Motor:

$$\mathbf{K_W.i} = \mathbf{J_W.W'} + \boldsymbol{\beta}_\mathbf{W}.\mathbf{W} + \boldsymbol{\tau}$$

onde: $\boldsymbol{\tau} = [\tau_D \;\; \tau_E]^T$

$$\mathbf{J_W} = \begin{bmatrix} J_D & 0 \\ 0 & J_E \end{bmatrix} \qquad \boldsymbol{\beta}_\mathbf{W} = \begin{bmatrix} \beta_D & 0 \\ 0 & \beta_E \end{bmatrix}$$

Substituindo a equação elétrica na equação mecânica do motor:

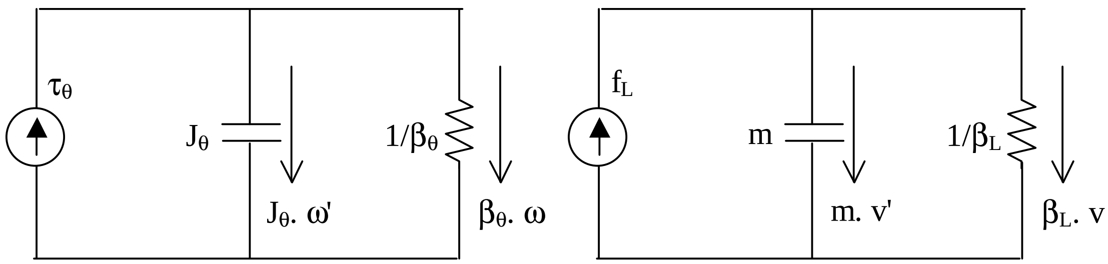
$$\boldsymbol{\tau} = -\,\mathbf{J_W.W'} - (\rho.\mathbf{K_W}^2 + \boldsymbol{\beta}_\mathbf{W}).\mathbf{W} + \rho.\mathbf{K_W.E}$$

lembrando que: $\mathbf{W} = {}^{\mathbf{W}}\mathbf{T_V.V}$ e $\mathbf{W'} = {}^{\mathbf{W}}\mathbf{T_V.V'}$,

$$\Rightarrow \quad \boldsymbol{\tau} = -\,\mathbf{J_W}.{}^{\mathbf{W}}\mathbf{T_V.V'} - (\rho.\mathbf{K_W}^2 + \boldsymbol{\beta}_\mathbf{W}).{}^{\mathbf{W}}\mathbf{T_V.V} + \rho.\mathbf{K_W.E}$$

- **Dinâmica do Robô**:

Equação Mecânica do Robô:



onde,

m = Massa do robô.
J_θ = Momento de inércia do robô.
β_L = Coeficiente de atrito das rodas em movimento linear.
β_θ = Coeficiente de atrito das rodas em movimento rotacional.

Lei de Newton: $f_L = m.v' + \beta_L.v$
Lei de Euler: $\tau_\theta = J_\theta.\omega' + \beta_\theta.\omega$

$\Rightarrow \quad \mathbf{F_V} = \mathbf{J_V}.\mathbf{V'} + \boldsymbol{\beta}_\mathbf{V}.\mathbf{V}$

onde: $\mathbf{F_V} = [f_L \;\; \tau_\theta]^T$

$$\mathbf{J_V} = \begin{bmatrix} m & 0 \\ 0 & J_\theta \end{bmatrix} \qquad \boldsymbol{\beta}_\mathbf{V} = \begin{bmatrix} \beta_L & 0 \\ 0 & \beta_\theta \end{bmatrix}$$

Em termos dos torques nas rodas ($\mathbf{F_V} = ({}^\mathbf{W}\mathbf{T_V})^T.\boldsymbol{\tau}$):

$\Rightarrow \quad ({}^\mathbf{W}\mathbf{T_V})^T.\boldsymbol{\tau} = \mathbf{J_V}.\mathbf{V'} + \boldsymbol{\beta}_\mathbf{V}.\mathbf{V}$

Multiplicando a dinâmica de atuadores por $({}^\mathbf{W}\mathbf{T_V})^T$:

$$({}^\mathbf{W}\mathbf{T_V})^T.\boldsymbol{\tau} = -[({}^\mathbf{W}\mathbf{T_V})^T.\mathbf{J_W}.\,{}^\mathbf{W}\mathbf{T_V}].\mathbf{V'} - [({}^\mathbf{W}\mathbf{T_V})^T.(\rho.\mathbf{K_W}^2 + \boldsymbol{\beta}_\mathbf{W}).\,{}^\mathbf{W}\mathbf{T_V}].\mathbf{V} + ({}^\mathbf{W}\mathbf{T_V})^T.\rho.\mathbf{K_W}.\mathbf{E}$$

Igualando as duas últimas equações e isolando o termo em **E**:

$$({}^\mathbf{W}\mathbf{T_V})^T.\rho.\mathbf{K_W}.\mathbf{E} = [\mathbf{J_V} + ({}^\mathbf{W}\mathbf{T_V})^T.\mathbf{J_W}.{}^\mathbf{W}\mathbf{T_V}].\mathbf{V'} + [\boldsymbol{\beta}_\mathbf{V} + ({}^\mathbf{W}\mathbf{T_V})^T.(\rho.\mathbf{K_W}^2 + \boldsymbol{\beta}_\mathbf{W}).{}^\mathbf{W}\mathbf{T_V}].\mathbf{V}$$

Fazendo:

$\mathbf{M_V} = [\mathbf{J_V} + ({}^\mathbf{W}\mathbf{T_V})^T.\mathbf{J_W}.{}^\mathbf{W}\mathbf{T_V}]$ = matriz de inércia, simétrica e definida positiva.

$\mathbf{B_V} = [\boldsymbol{\beta}_\mathbf{V} + ({}^\mathbf{W}\mathbf{T_V})^T.(\rho.\mathbf{K_W}^2 + \boldsymbol{\beta}_\mathbf{W}).\,{}^\mathbf{W}\mathbf{T_V}]$ = matriz de coeficientes de atritos viscosos, simétrica e definida positiva. Fazendo:

$\boldsymbol{\tau}_\mathbf{V} = ({}^\mathbf{W}\mathbf{T_V})^T.\rho.\mathbf{K_W}.\mathbf{E} \quad \Rightarrow \quad \mathbf{E} = [({}^\mathbf{W}\mathbf{T_V})^T.\rho.\mathbf{K_W}]^{-1}.\boldsymbol{\tau}_\mathbf{V}$

Modelo Dinâmico em V: $\boldsymbol{\tau}_\mathbf{V} = \mathbf{M_V}.\mathbf{V'} + \mathbf{B_V}.\mathbf{V}$

Substituindo $\mathbf{V} = ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}.\mathbf{q'}$ e $\mathbf{V'} = ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}.\mathbf{q''}$:

$$\tau_{\mathrm{V}} = \mathbf{M}_{\mathrm{V}}.\ ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}.\mathbf{q''} + \mathbf{B}_{\mathrm{V}}.\ ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}.\mathbf{q'}$$

Multiplicando os dois lados por ${}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}$:

$${}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\tau_{\mathrm{V}} = [{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{M}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}].\mathbf{q''} + [{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{B}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}].\mathbf{q'}$$

Chamando:

$\mathbf{M}_{\mathbf{q}} = [{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{M}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}]$ = Matriz de inércia em espaço de configuração, simétrica e definida positiva.

$\mathbf{B}_{\mathbf{q}} = [{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{B}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}]$ = Matriz de coeficientes de atritos viscosos em espaço de configuração, simétrica e definida positiva.

$$\tau_{\mathbf{q}} = {}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\tau_{\mathrm{V}} \quad \Rightarrow \quad \tau_{\mathrm{V}} = ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}.\tau_{\mathbf{q}}$$

Modelo Dinâmico em q: $\tau_{\mathbf{q}} = \mathbf{M}_{\mathbf{q}}.\mathbf{q''} + \mathbf{B}_{\mathbf{q}}.\mathbf{q'}$

- **Propriedade da Matriz de Inércia:**

$$\mathbf{M}_{\mathbf{q}}' = {}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{M}_{\mathrm{V}}.(({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}})' + ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})'.\mathbf{M}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}$$

Dado um vetor **e**, tomando a forma quadrática $[\mathbf{e}^{\mathbf{T}}.\mathbf{M}_{\mathbf{q}}'.\mathbf{e}]/2$:

$$[\mathbf{e}^{\mathbf{T}}.\mathbf{M}_{\mathbf{q}}'.\mathbf{e}]/2 = \mathbf{e}^{\mathbf{T}}.[{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{M}_{\mathrm{V}}.(({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}})' + ({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})'.\mathbf{M}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}].\mathbf{e}/2 =$$

$$= \mathbf{e}^{\mathbf{T}}.[{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{M}_{\mathrm{V}}.(({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}})'].\mathbf{e}/2 + \mathbf{e}^{\mathbf{T}}.[({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})'.\mathbf{M}_{\mathrm{V}}.({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}}].\mathbf{e}/2 =$$

$$= \mathbf{e}^{\mathbf{T}}.[{}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}}.\mathbf{M}_{\mathrm{V}}.(({}^{\mathbf{q}}\mathbf{T}_{\mathrm{V}})^{\mathrm{T}})'].\mathbf{e}$$

- **Forma Linear em Parâmetros:**

  Modelo Dinâmico em V Linear em Parâmetros:

$$\boldsymbol{\tau_V} = \mathbf{M_V}.\mathbf{V'} + \mathbf{B_V}.\mathbf{V}$$

$$= \begin{bmatrix} M_{V11} & M_{V12} \\ M_{V12} & M_{V22} \end{bmatrix} . \begin{bmatrix} v' \\ \omega' \end{bmatrix} + \begin{bmatrix} B_{V11} & B_{V12} \\ B_{V12} & B_{V22} \end{bmatrix} . \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} M_{V11}.v'+M_{V12}.\omega' \\ M_{V12}.v'+M_{V22}.\omega' \end{bmatrix} + \begin{bmatrix} B_{V11}.v+B_{V12}.\omega \\ B_{V12}.v+B_{V22}.\omega \end{bmatrix}$$

$$= \begin{bmatrix} v' & \omega' & 0 \\ 0 & v' & \omega' \end{bmatrix} . \begin{bmatrix} M_{V11} \\ M_{V12} \\ M_{V22} \end{bmatrix} + \begin{bmatrix} v & \omega & 0 \\ 0 & v & \omega \end{bmatrix} . \begin{bmatrix} B_{V11} \\ B_{V12} \\ B_{V22} \end{bmatrix}$$

$$= \boldsymbol{\Phi_{Vm}}.\mathbf{m} + \boldsymbol{\Phi_{Vb}}.\mathbf{b} = \boldsymbol{\Phi_V}.\mathbf{P}$$

onde,

$$\boldsymbol{\Phi_{Vm}} = \begin{bmatrix} v' & \omega' & 0 \\ 0 & v' & \omega' \end{bmatrix} \qquad \boldsymbol{\Phi_{Vb}} = \begin{bmatrix} v & \omega & 0 \\ 0 & v & \omega \end{bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} M_{V11} \\ M_{V12} \\ M_{V22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} B_{V11} \\ B_{V12} \\ B_{V22} \end{bmatrix}$$

$$\boldsymbol{\Phi_V} = [\boldsymbol{\Phi_{Vm}} \quad \boldsymbol{\Phi_{Vb}}]$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{m} \\ \mathbf{b} \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{\tau_V} = \boldsymbol{\Phi_V}.\mathbf{P}$$

Modelo Dinâmico em q Linear em Parâmetros:

Multiplicando o modelo em **V** pela matriz $^{q}\mathbf{T}_{V}$:

$^{q}\mathbf{T}_{V}.\boldsymbol{\tau}_{V} = {}^{q}\mathbf{T}_{V}.\boldsymbol{\Phi}_{V}.\mathbf{P} \quad \Rightarrow \quad \boldsymbol{\tau}_{q} = \boldsymbol{\Phi}_{q}.\mathbf{P}$

onde: $\quad \boldsymbol{\tau}_{q} = {}^{q}\mathbf{T}_{V}.\boldsymbol{\tau}_{V} \qquad \boldsymbol{\Phi}_{q} = {}^{q}\mathbf{T}_{V}.\boldsymbol{\Phi}_{V}$

$\Rightarrow \quad \boldsymbol{\Phi}_{q} = [{}^{q}\mathbf{T}_{V}.\boldsymbol{\Phi}_{Vm} \quad {}^{q}\mathbf{T}_{V}.\boldsymbol{\Phi}_{Vb}] = [\boldsymbol{\Phi}_{qm} \quad \boldsymbol{\Phi}_{qb}]$

onde:

$\boldsymbol{\Phi}_{qm} = {}^{q}\mathbf{T}_{V}.\boldsymbol{\Phi}_{Vm}$

$\boldsymbol{\Phi}_{qb} = {}^{q}\mathbf{T}_{V}.\boldsymbol{\Phi}_{Vb}$

As matrizes $\boldsymbol{\Phi}_{Vm}$ e $\boldsymbol{\Phi}_{Vb}$ podem ser expressas como:

$\boldsymbol{\Phi}_{Vm} = [(\mathbf{I}^{11}.\mathbf{V}') \quad (\mathbf{I}^{12}.\mathbf{V}') \quad (\mathbf{I}^{22}.\mathbf{V}')]$

$\boldsymbol{\Phi}_{Vb} = [(\mathbf{I}^{11}.\mathbf{V}) \quad (\mathbf{I}^{12}.\mathbf{V}) \quad (\mathbf{I}^{22}.\mathbf{V})]$

onde:

$$\mathbf{I}^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{I}^{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{I}^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Assim, lembrando que $\mathbf{V} = ({}^{q}\mathbf{T}_{V})^{T}.\mathbf{q}'$ e $\mathbf{V}' = ({}^{q}\mathbf{T}_{V})^{T}.\mathbf{q}''$:

$\boldsymbol{\Phi}_{qm} = [[{}^{q}\mathbf{T}_{V}\mathbf{I}^{11}({}^{q}\mathbf{T}_{V})^{T}\mathbf{q}''] \quad [{}^{q}\mathbf{T}_{V}\mathbf{I}^{12}({}^{q}\mathbf{T}_{V})^{T}\mathbf{q}''] \quad [{}^{q}\mathbf{T}_{V}\mathbf{I}^{22}({}^{q}\mathbf{T}_{V})^{T}\mathbf{q}'']]$

$\boldsymbol{\Phi}_{qb} = [[{}^{q}\mathbf{T}_{V}\mathbf{I}^{11}({}^{q}\mathbf{T}_{V})^{T}\mathbf{q}'] \quad [{}^{q}\mathbf{T}_{V}\mathbf{I}^{12}({}^{q}\mathbf{T}_{V})^{T}\mathbf{q}'] \quad [{}^{q}\mathbf{T}_{V}\mathbf{I}^{22}({}^{q}\mathbf{T}_{V})^{T}\mathbf{q}']]$

- **Modelo Dinâmico Completo:**

Lembrando que, do modelo dinâmico em **V**, temos:

$\tau_V = \mathbf{M_V}.\mathbf{V'} + \mathbf{B_V}.\mathbf{V}$ e $\tau_V = ({}^{W}\mathbf{T_V})^T.\rho.\mathbf{K_W}.\mathbf{E}$

$\Rightarrow \quad \mathbf{V'} = (\mathbf{M_V})^{-1}.[-\mathbf{B_V}.\mathbf{V} + ({}^{W}\mathbf{T_V})^T.\rho.\mathbf{K_W}.\mathbf{E}]$

Lembrando que o modelo cinemático é dado por:

$\Rightarrow \quad \mathbf{q'} = [{}^{q}\mathbf{T_V}].\mathbf{V}$

Agrupando as duas equações acima obtemos o modelo dinâmico completo, incluindo dinâmica de atuadores, para o robô móvel:

Equação de Estado:

$$\begin{bmatrix} \mathbf{V'} \\ \mathbf{q'} \end{bmatrix} = \begin{bmatrix} [-\mathbf{M_V}^{-1}.\mathbf{B_V}] & \mathbf{0} \\ [{}^{q}\mathbf{T_V}] & \mathbf{0} \end{bmatrix} . \begin{bmatrix} \mathbf{V} \\ \mathbf{q} \end{bmatrix} + \begin{bmatrix} [\mathbf{M_V}^{-1}.\ {}^{W}\mathbf{T_V}^T.\rho.\mathbf{K_W}] \\ \mathbf{0} \end{bmatrix} . \begin{bmatrix} \mathbf{E} \end{bmatrix}$$

Equação de Saída:

$$\begin{bmatrix} \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} . \begin{bmatrix} \mathbf{V} \\ \mathbf{q} \end{bmatrix}$$

Características do Modelo:

- Modelo de quinta ordem.
- Não linear (cos(θ) e sen(θ) na matriz ${}^{q}\mathbf{T_V}$).
- Sistema MIMO (Vetor de entradas **E** / Vetor de saídas, **q**)
- Sistema subatuado (duas entradas, **E** para três saídas, **q**).