Spatial Properties of the Image Function

Dirac delta operator:

$$\delta(x-\xi) = \begin{cases} \infty & x = \xi \\ 0 & otherwise \end{cases}$$

This function has the following properties:

$$\int_{-\epsilon}^{\epsilon} \delta(x)dx \; = \; 1 \;\; for \; \epsilon > 0$$

$$\int_{-\infty}^{\infty} F(\xi)\delta(x-\xi)d\xi \; = \; F(x) \qquad \textbf{Sifting Property}$$

Spatial properties of the image function

Fourier Transform

$$\mathcal{F}[f(x,y)] = F(u,v) = \iint_{-\infty}^{\infty} f(x,y) exp\{-i(2\pi(ux+vy)\} dxdy$$

where u is the spatial frequency (in $cycles/pixel$), so that when x is specified in pixels, $(2\pi ux)$ is in radians, and $i = \sqrt{-1}$.

$$\mathcal{F}^{-1}[F(u,v)] = f(x,y) = \iint_{-\infty}^{\infty} F(u,v) exp\{i(2\pi(ux+vy))\} dudv$$

Fourier Transform Pairs

Table 1: Fourier Transform Pairs

$$F(\omega) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad \omega(rad/pixel) = 2\pi u(cycles/pixel)$$

Name	$f(x)$	$F(\omega)$
rectangular function	$rect(x) = 1 \quad -\frac{1}{2} < x < \frac{1}{2}$	$sinc(\omega/2\pi) = \frac{sin(\omega/2)}{\omega/2}$
triangular function	$tri(x) = 2(x + \frac{1}{2}) \quad -\frac{1}{2} < x < 0$ $1 - 2(x) \quad 0 < x < \frac{1}{2}$	$sinc^2(\omega/2\pi)$
Gaussian	$e^{-\alpha\|x\|}$ e^{-px^2}	$2\alpha/(\alpha^2 + \omega^2)$ $\frac{1}{\sqrt{2p}} e^{-\omega^2/4p}$
unit impulse	$\delta(x)$	1
comb function	$\sum_n \delta(x - nx_0)$	$\frac{1}{x_0} \sum_n \delta(\frac{\omega}{2\pi} - \frac{n}{x_0})$
differentiation	$g^n(x)$	$(i\omega)^n G(\omega)$
linear combination	$ag(x) + bh(x)$	$aG(\omega) + bH(\omega)$
scale	$f(ax)$	$\frac{1}{\|a\|} F(\frac{\omega}{a})$

Shift and Convolution Theorems

Shift Theorem:

$$
\begin{aligned}
\mathcal{F}[f(x)] &= \int_{-\infty}^{\infty} f(x) exp\{-i(2\pi ux)\} dx, \ then \\
\mathcal{F}[f(x-a)] &= \int_{-\infty}^{\infty} f(x-a) exp\{-i(2\pi ux)\} dx, \ then \\
&= \int_{-\infty}^{\infty} f(x') exp\{-i(2\pi u(x'+a))\} dx', \ and \\
&= exp\{-i(2\pi ua)\} \int_{-\infty}^{\infty} f(x') exp\{-i(2\pi ux')\} dx',
\end{aligned}
$$

so that

$$
\mathcal{F}[f(x-a)] = exp\{-i(2\pi ua)\} \mathcal{F}(f(x))
$$

Shift and Convolution Theorems

Convolution Theorem:

The convolution of two functions $f(x)$ and $g(x)$:

$$f(x) * g(x) = h(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

where α an integration variable.

$$\begin{aligned}
\mathcal{F}[f(x) * g(x)] &= \mathcal{F}[h(x)] \\
&= \mathcal{F}\left[\int_\alpha f(\alpha) g(x-\alpha) d\alpha\right] \\
&= \int_x \left[\int_\alpha f(\alpha) g(x-\alpha) d\alpha\right] exp\{-i2\pi ux\} dx \\
&= \int_\alpha f(\alpha) \left[\int_x g(x-\alpha) exp\{-i(2\pi ux)\} dx\right] d\alpha
\end{aligned}$$

and by the Shift theorem,

$$= \int_\alpha f(\alpha) exp\{-i(2\pi u\alpha)\} d\alpha \int_x g(x) exp\{-i(2\pi ux)\} dx,$$

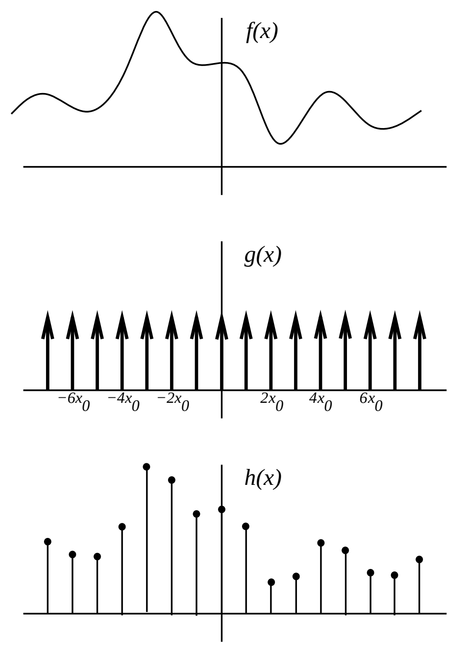
therefore,

$$\mathcal{F}[f(x) * g(x)] = F(u)G(u)$$

therefore, spatial convolution $\Longleftrightarrow$ frequency domain multiplication

Sampling Theorem - Aliasing

A continuous spatial function, $f(x)$ is *sampled* by computing the product of $f(x)$ and $g(x)$, an infinite sequence of Dirac delta operators.



$$
\begin{aligned}
h(x) &= f(x)\sum_n \delta(x - nx_0) \\
&= \sum_n f(nx_0)\delta(x - nx_0)
\end{aligned}
$$

By the convolution theorem, we know that the product of these two spatial functions is equivalent to the convolution of their Fourier transform pairs.

$$
\begin{aligned}
f(x) &\xrightarrow{\mathcal{F}} F(u) \\
\sum_n \delta(x - nx_0) &\xrightarrow{\mathcal{F}} \frac{1}{x_0}\sum_n \delta(u - \frac{n}{x_0}) = G(u),\ \textit{and} \\
H(u) &= F(u) * G(u)
\end{aligned}
$$

Sampling Theorem - Aliasing (cont.)

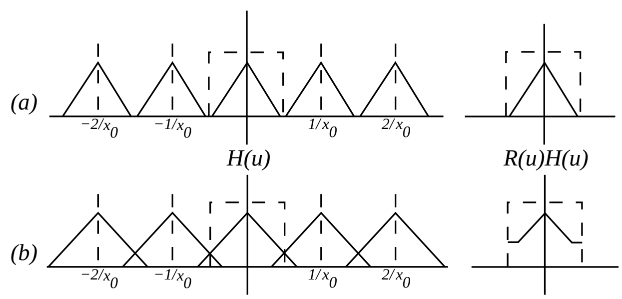
$$H(u) = F(u) * G(u)$$

We may write the function $H(u)$ in terms of $F(u)$:

$$\begin{aligned}
H(u) &= F(u)*\left[\frac{1}{x_0}\sum_n \delta(u-\frac{n}{x_0})\right] \\
&= \int_{-\infty}^{\infty} F(\alpha)G(u-\alpha)d\alpha \\
&= \int_{-\infty}^{\infty} F(\alpha)\left[\frac{1}{x_0}\sum_n \delta(u-\alpha-\frac{n}{x_0})d\alpha\right] \\
&= \frac{1}{x_0}\int_{-\infty}^{\infty}\sum_n F(u-\frac{n}{x_0})\delta(u-\alpha-\frac{n}{x_0})d\alpha \\
&= \frac{1}{x_0}\sum_n F(u-\frac{n}{x_0})\int_{-\infty}^{\infty}\delta(u-\alpha-\frac{n}{x_0})d\alpha \\
H(u) &= \frac{1}{x_0}\sum_n F(u-\frac{n}{x_0})
\end{aligned}$$

Therefore, the frequency spectrum of the sampled image consists of duplicates of the spectrum of the original image distributed at $1/x_0$ frequency intervals.

Aliasing (cont.)



$R(u)$ is a frequency domain bandpass filter

$$R(u) = \begin{array}{l} 1 \;\; if \;\; |u| < 1/(2x_0), \\ 0 \;\; otherwise \end{array}$$

Aliasing: When replicated spectra interfere, the crosstalk introduces energy at relatively high frequencies changing the appearance of the reconstructed image.

The Sampling Theorem: *if the image contains no frequency components greater than one half the sampling frequency, then the continuous image is faithfully represented in the sampled image.*

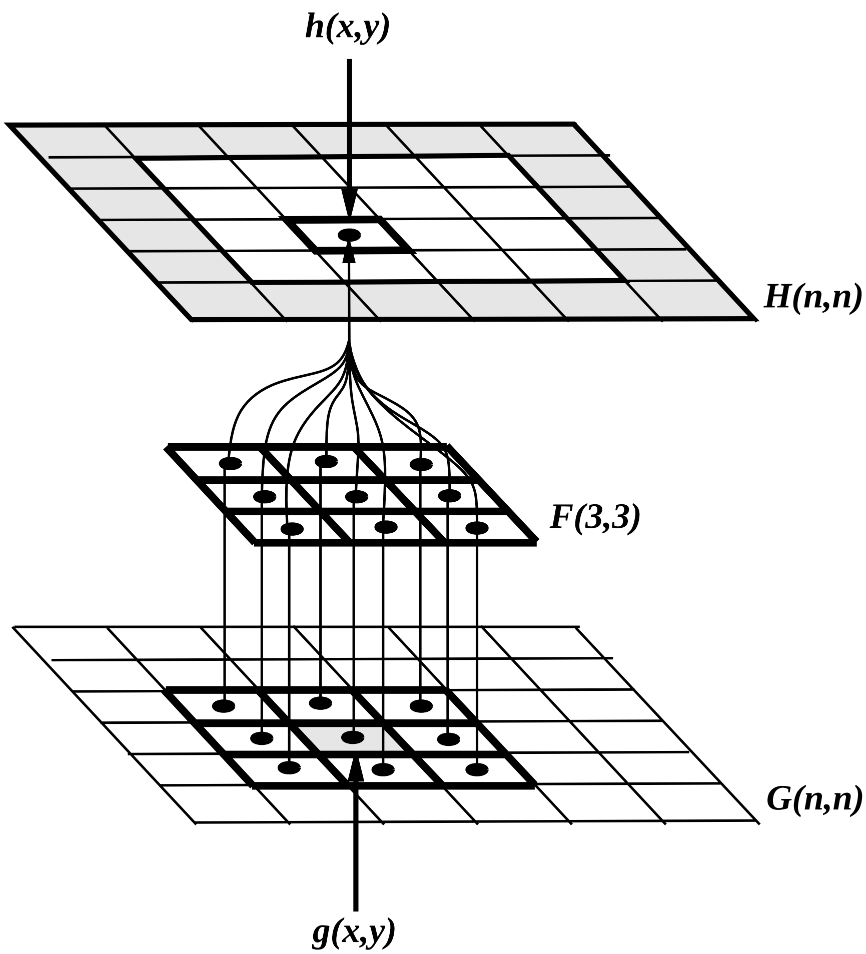
Early Processing - Convolution

The convolution of two functions $f(x)$ and $g(x)$:

$$f(x,y) * g(x,y) = h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)g(x-u,y-v)dudv$$

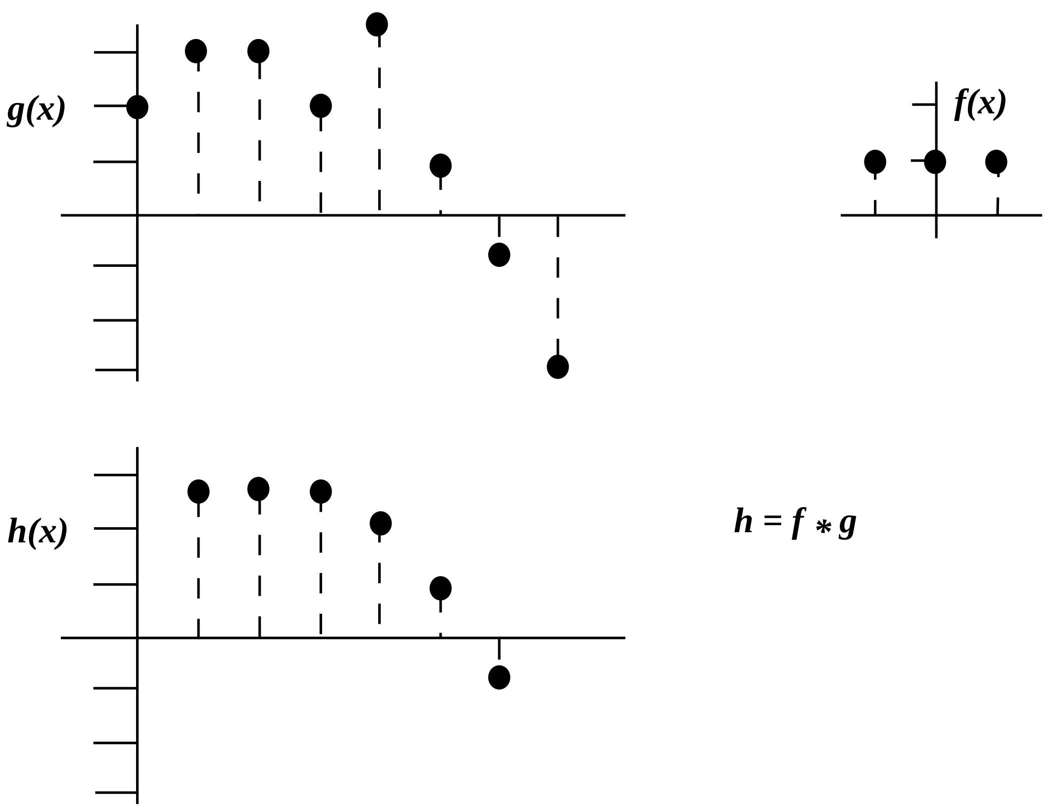
or

$$f(x,y) * g(x,y) = h(x,y) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f(u,v)g(x-u,y-v)$$



Early Processing - Smoothing

Low Pass Filter



Early Processing - Edges

Intensity Gradients

$$\nabla g = \frac{dg}{dx}\hat{x} + \frac{dg}{dy}\hat{y}$$

$$\frac{dg(x,y)}{dx} \approx \frac{g(x+1,y) - g(x-1,y)}{2}$$

$$\Rightarrow f_x = \left[-\frac{1}{2},\ 0,\ \frac{1}{2}\right]_{1x3}$$

$$\frac{dg(x,y)}{dy} \approx \frac{g(x,y+1) - g(x,y-1)}{2}$$

$$\Rightarrow f_y = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}_{3x1}$$

$$\begin{aligned} |\nabla g| &= \left[\left(\frac{dg}{dx}\right)^2 + \left(\frac{dg}{dy}\right)^2\right]^{\frac{1}{2}} \\ &= \\ \phi &= tan^{-1}\left(\frac{dg/dy}{dg/dx}\right) \end{aligned}$$

Edge Operators

operator	∇_1	∇_2
Roberts	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Prewit	$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$
Sobel	$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

Early Processing - Edge Sharpening

Laplacian

$$\nabla^2 g = \frac{d^2 g}{dx^2} + \frac{d^2 g}{dy^2} \Rightarrow f = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$