The Jacobian in the Force Domain

$$\text{Work out} = \text{Work in}$$

$$F^T \Delta x = \tau^T \Delta \theta$$

$$\Delta x = J \Delta \theta$$

$$\text{therefore: } F^T [J \Delta \theta] = \tau^T \Delta \theta$$

$$F^T J = \tau^T,$$

$$\text{or } \tau = J^T F$$

Review — Eigenvalues and Eigenvectors

$$\vec{y} = A\vec{x}$$

if A is a real $n \times n$ matrix, the polynomial

$$p(\lambda) = det(A - \lambda I)$$

is the *characteristic polynomial* of A.

for a root of the characteristic polynomial, λ^*,

$$(A - \lambda^* I)\vec{x}^* = \vec{0},\ \vec{x}^* \neq \vec{0}$$

$p(\lambda)$: roots λ^* are **eigenvalues**

$\vec{x}^*$ are the **eigenvectors**

Review — EXAMPLE

Suppose

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$det(A - \lambda I) = (2-\lambda)(\lambda^2 - \lambda + 1)$$

$$\Rightarrow \lambda = 2, \frac{1}{2} \pm (\frac{\sqrt{3}}{2})i$$

for

$$\lambda = 2 \Rightarrow (A - \lambda I): \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix} \vec{x} = \vec{0}$$

$$\begin{array}{rcrcl} & -x_1 & + & x_3 & = & 0 \\ \Rightarrow & 2x_1 & + & x_3 & = & 0 \\ & -x_1 & - & 2x_3 & = & 0 \end{array} \Rightarrow x_1 = x_3 = 0, x_2 = 1$$

The Manipulator Jacobian: Principle Kinematic Transformations

"...posture variation is a means through which motion and strength characteristics of the arm is made compatible with the task [**Chiu87**]."

The Jacobian transforms the unit sphere in joint velocity space:

$$\|\dot{\theta}\|^2 = \dot{\theta}_0^2 + \dot{\theta}_1^2 + \ldots + \dot{\theta}_n^2 \leq 1$$

to an ellipsoid in Cartesian space, since $\dot{x} = J\dot{\theta}$, and

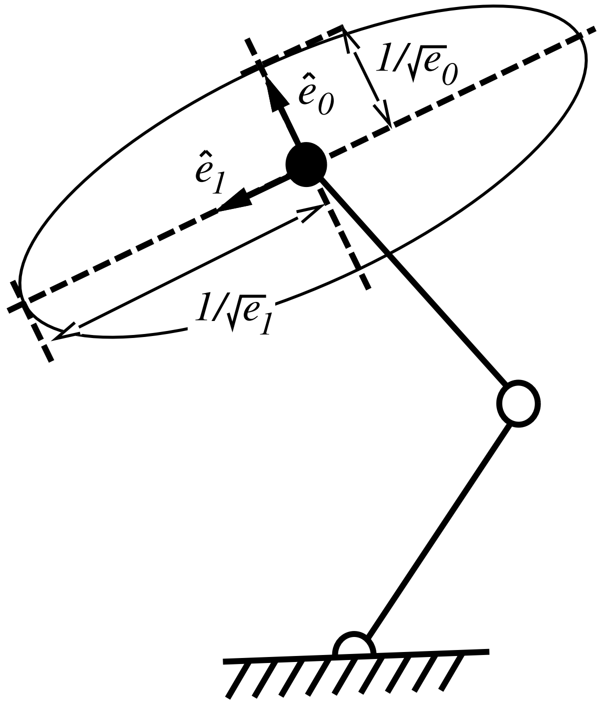
$$\dot{\theta}^T\dot{\theta} = \|\dot{\theta}\|^2 = (J^{-1}\dot{x})^T(J^{-1}\dot{x}) = \dot{x}^T[(J^{-1})^T J^{-1}]\dot{x} = \dot{x}^T(JJ^T)^{-1}\dot{x}.$$

$$\begin{array}{c} \textit{principle directions} \\ \textit{of the transformation} \end{array} \Leftrightarrow \textit{eigenvectors } (JJ^T)^{-1}$$

$$\textit{amplification/attenuation} \Leftrightarrow \textit{eigenvalues } (JJ^T)^{-1}$$

Jacobian — Conditioning Metrics

Principle Kinematic Axes — cont.



The velocity ellipsoid derived from (JJ^T)

$$AMPLIFICATION \Leftrightarrow PRECISION$$

Review — Matrix Norms

any real-valued function s.t.

1. $||A|| \geq 0$
2. $||A|| = 0$ iff $A = [0]$ identically
3. $||\alpha A|| = |\alpha| ||A||$
4. $||A + B|| \leq ||A|| + ||B||$
5. $||AB|| \leq ||A|| ||B||$

EXAMPLE:

$$||A||_\infty = max_i \sum_{j=1}^{n} |a_{ij}|$$

Jacobian — Conditioning Metrics

Condition Number

$$A\vec{x} = \vec{b}$$

let $\tilde{x}$ be an approximate solution,

$$\Rightarrow \quad d\vec{b} = \vec{b} - A\tilde{x}$$

$$d\vec{b} = \vec{b} - A\tilde{x} = A\vec{x} - A\tilde{x} \quad \Rightarrow \quad \vec{x} - \tilde{x} = A^{-1}d\vec{b}$$

therefore,

$$||\vec{x} - \tilde{x}|| = ||A^{-1}d\vec{b}|| \leq ||A^{-1}||\ ||d\vec{b}||$$

since,

$$\vec{b} = A\vec{x} \qquad \begin{aligned} ||\vec{b}|| &\leq ||A||||\vec{x}|| \\ ||\vec{x}|| &\geq \frac{||\vec{b}||}{||A||} \end{aligned}$$

and

$$\frac{||\vec{x} - \tilde{x}||}{||\vec{x}||} \leq \frac{||A^{-1}||\ ||d\vec{b}||}{||\vec{b}||/||A||} = ||A||\ ||A^{-1}||\ \frac{||d\vec{b}||}{||\vec{b}||}$$

Condition Number - cont.

$$\frac{||d\vec{x}||}{||x||} = \kappa(A)\frac{||d\vec{b}||}{||b||} \qquad \kappa(A) = ||A||\,||A^{-1}||$$

the *condition number* describes the error amplification capacity of the transform A,

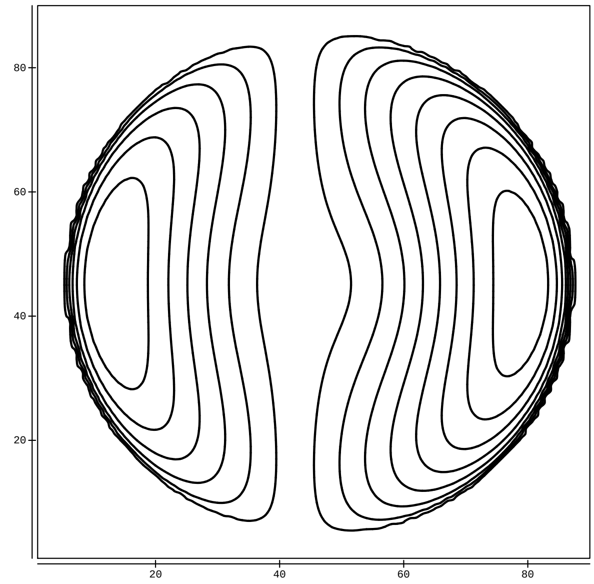
$$1 \leq \kappa(A) \leq \infty$$

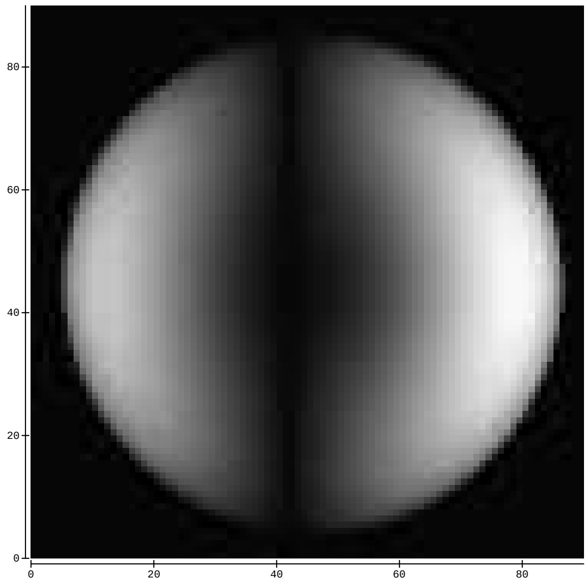
"well-conditioned"
"isotropic"
$small\ d\vec{b} \;\Leftrightarrow\; small\ d\vec{x}$

"ill-conditioned"

Manipulability

$$\kappa(J) = \sqrt{det(JJ^T)}$$

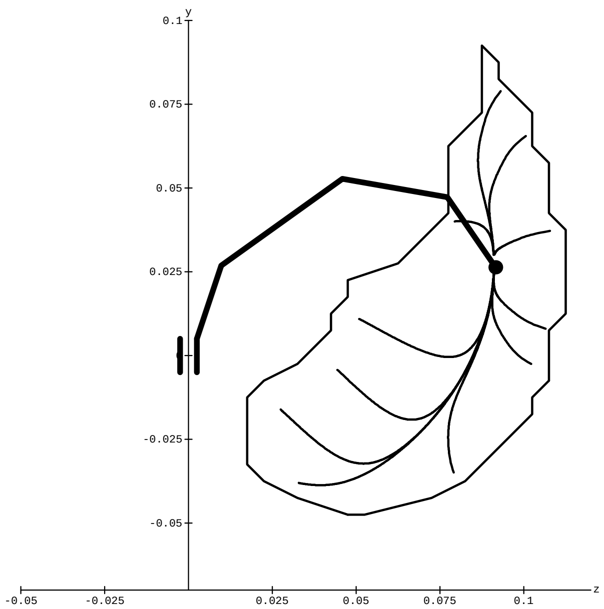




Manipulability: cont.

$$\frac{dM}{d\vec{\theta}_{arm}} = \underbrace{\frac{\partial M}{\partial \vec{x}}}_{gradient\ field} \ {}_f\mathbf{R}_{arm} \ \underbrace{\frac{\partial \vec{X}}{\partial \theta_{arm}}}_{arm\ Jacobian}$$

$$\Delta\vec{\theta}_{arm} = K_m * \frac{dM}{d\vec{\theta}_{arm}}$$



Inverting Non-Square Jacobians — Redundant Manipulators

Pseudoinverse

$$\vec{\rho} = \dot{\vec{x}}_{nx1} - J_{nxm}\dot{\theta}_{mx1}$$
$$E = \vec{\rho}^T \vec{\rho} = (\dot{\vec{x}} - J\dot{\theta})^T (\dot{\vec{x}} - J\dot{\theta})$$

this quadratic construction is positive semi-definite, $\frac{\partial^2 E}{\partial \dot{\vec{\theta}}^2} \geq 0$, and therefore has a unique minimum...

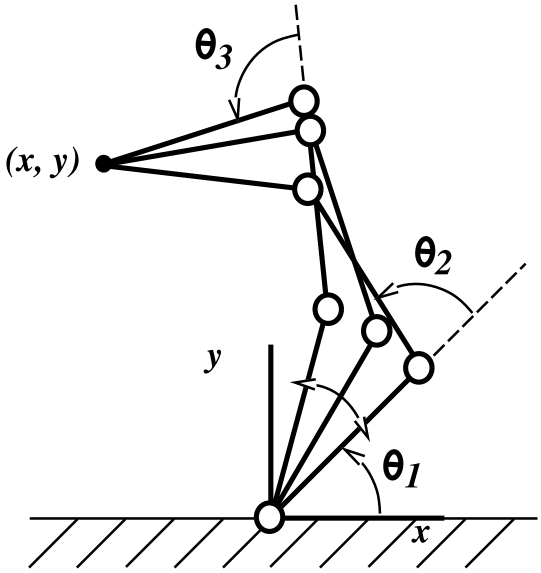
$$\frac{\partial E}{\partial \dot{\vec{\theta}}} = 0 = J^T(\dot{\vec{x}} - J\dot{\vec{\theta}}),\ or$$

$$J^T \dot{\vec{x}} = J^T J \dot{\vec{\theta}},\ and$$

$$\begin{aligned} \dot{\vec{\theta}} &= \left[J^T J\right]^{-1} J^T \dot{\vec{x}} \qquad (m < n) \\ &= J^T \left[J J^T\right]^{-1} \dot{\vec{x}} \qquad (m \geq n) \end{aligned}$$

$$\dot{\vec{\theta}} = J^+ \dot{\vec{x}}$$

Redundant Manipulators - Self-Motion Manifolds



$$
\begin{aligned}
x &= l_1 cos(\theta_1) \;+ l_2 cos(\theta_1+\theta_2) \\
&\qquad\qquad\quad + l_3 cos(\theta_1+\theta_2+\theta_3) \\
y &= l_1 sin(\theta_1) \;+ l_2 sin(\theta_1+\theta_2) \\
&\qquad\qquad\quad + l_3 sin(\theta_1+\theta_2+\theta_3)
\end{aligned}
$$

therefore, the *self-motion manifold* is defined by:

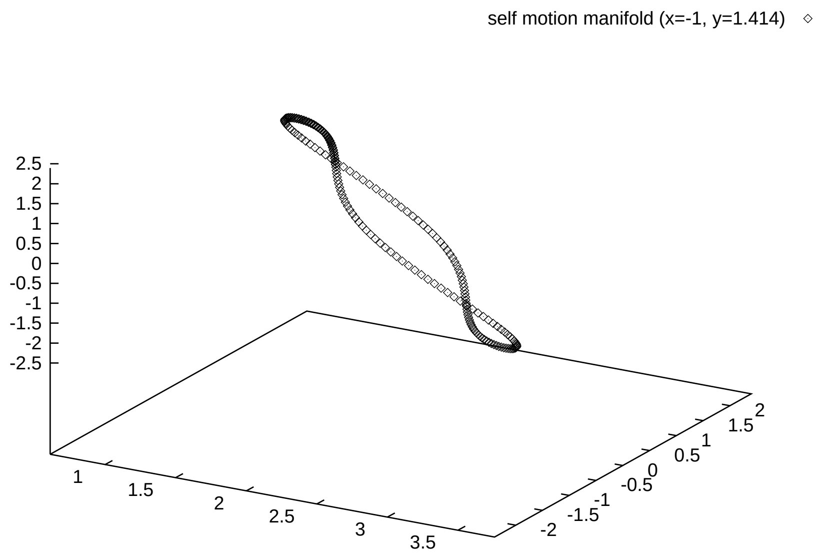
$$
\begin{aligned}
\left\{ \begin{array}{c} \dot{x} \\ \dot{y} \end{array} \right\} &= \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = [J] \left\{ \begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{array} \right\} \\
&= \left[\begin{array}{ccc} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \end{array} \right] \left\{ \begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{array} \right\}
\end{aligned}
$$

Redundant Manipulators - Self-Motion Manifolds (cont.)

For this case, internal motions along the self-motion manifold are constrained to be in the *null space* of the manipulator Jacobian:

$$\dot{\vec{\theta}}_{null} = \left\{ \begin{array}{c} l_2 l_3 sin(\theta_3) \\ -l_2 l_3 sin(\theta_3) - l_1 l_3 sin(\theta_2 + \theta_3) \\ l_1 l_2 sin(\theta_2) + l_1 l_3 sin(\theta_2 + \theta_3) \end{array} \right\}$$

If we follow this null space vector through a sequence of configurations, we can generate the self-motion manifold. A motion along the self-motion manifold is called an *internal motion*.



Redundant Manipulators

Pseudoinverse — secondary performance criteria

$$\dot{\theta} = J^{+}\dot{x} + (I - J^{+}J)\kappa$$

If κ is a $\dot{\vec{\theta}}$ command that addresses a secondary performance criteria:

$$\kappa = K\frac{\partial p}{\partial \theta}.$$

then $(I - J^{+}J)\kappa$ projects this velocity command onto the *null* space of J.

manipulability $p = \sqrt{detJJ^{T}}$

postural bias $p = (\theta - \theta_{ref})^{T}H(\theta - \theta_{ref})$

task compatibility $p = M^{T}M$, where $M = \sum_{i=1}^{m} w_i \frac{(T_{i,act} - T_{i,des})}{T_{i,des}}$
where T_i represents the transmission ratio along a task specified direction, u_i.

Redundant Manipulators — cont.

SR-Inverse

The pseudoinverse minimizes the squared Euclidean error

$$E = (\partial x - J\partial\theta)^T(\partial x - J\partial\theta)$$

and so explicitly addresses the "exactness" of the inverse solution. However, when the manipulator is near a singular configuration, the solution may still call for exceedingly large joint angle velocities.

The SR-inverse minimizes the weighted sum of "exactness" and "feasibility" errors

$$E = w_1 * \|\partial x - J\partial\theta\|^2 + w_2 * \|\partial\theta\|^2$$

where w_1 and w_2 are weights expressing the relative importance of exactness and feasibility, respectively. The result is:

$$J^* = (J^T J + KI)^{-1} J^T = J^T(JJ^T + kI)^{-1}$$

where k is the ratio w_2/w_1 ($k = 0$ yields the pseudoinverse).